1.7(900) **Einige Identitäten, Ungleichungen und Definitionen**

(900)**Idenditäten:**

1.) $\sum_{k=m}^{n} a_k = \sum_{\nu=m}^{n} a_\nu$ 2.) $\sum_{k=m}^{n} a_k = \sum_{k=m-\ell}^{n-\ell} a_{k+}$ ($\in \mathbb{Z}$) 3.) $\sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n$

4.) $\sum_{k=m}^{n} a_k = \sum_{\nu=m}^{n} a_{n+m-\nu}$ Kommutativgesetz

zur Erläuterung:

ν	m	m+1 ..... n-1		n
n+m−ν	n	n−1	m+1	m

5.) $\sum_{k=m}^{n} (ca_k) = c\sum_{k=m}^{n} a_k$ Distributivgesetz

6.) $\sum_{k=m}^{n} 1 = n-m+1$ falls $n \geq m$ (Bsp $\sum_{k=5}^{6} 1 = 6-5+1=2$)

7.) $\sum_{k=m}^{n} (a_k \pm b_k) = (\sum_{k=m}^{n} a_k) \pm (\sum_{k=m}^{n} b_k)$, $n \geq m$,

8.) $m,n \in \mathbb{Z}$, $\forall\ a_k \in \mathbb{C}$, $k \geq m$

(.) $|\sum_{k=m}^{n} a_k| \leq \sum_{k=m}^{n} |a_k|$ ($n \geq m$)(Dreiecksungleichung)

*//**S1.2.1** (406) Vor: K angeordnet, a,b∈K 6.)|a+b|≤|a|+|b| Δ Ungl//*

Bew: $n \leq m$ trivial 0=0

$n \geq m$, Induktion nach n, $n_0 = m$

$n=n_0=m$: $|\sum_{k=m}^{m} a_k| = |a_k| = \sum_{k=m}^{m} |a_k|$

$n \mapsto n+1$: $|\sum_{k=m}^{n+1} a_k| = |\sum_{k=m}^{n} a_k + a_{n+1}| \underbrace{\geq}_{S1.2.1} \sum_{k=m}^{n} |a_k| + |a_{n+1}| = \sum_{k=m}^{n+1} |a_k|$

(..) $|\sum_{k=m}^{n} a_k| \geq |a_m| - \sum_{k=m+1}^{n} |a_k|$ ($n \geq m$)

*//**S1.2.1** (406) Vor: K angeordnet, a,b∈K 7.)|a+b|≥|a|−|b| Δ Ungl//*

Bew: $|\sum_{k=m}^{n} a_k| = |\sum_{k=m+1}^{n} a_k + a_m| \underbrace{\geq}_{S1.2.1} |a_m| - |\sum_{k=m+1}^{n} a_k| \geq |a_m| - |\sum_{k=m+1}^{n} a_k|$

9.) $\sum_{\nu=m_1}^{n_1}\sum_{m=m_2}^{n_2} a_{\nu\mu} = \sum_{m=m_2}^{n_2}\sum_{\nu=m_1}^{n_1} a_{\nu\mu}$, $\sum_{\nu=1}^{n}\sum_{m=1}^{\nu} a_{\nu\mu} = \sum_{m=1}^{n}\sum_{\nu=m}^{n} a_{\nu\mu}$

$\sum_{\nu=m_1}^{n_1}\sum_{m=m_2}^{n_2} a_{\nu\mu} = \sum_{\nu=m_1}^{n_1} b_\nu = \sum_{\mu=m_2}^{n_2}\sum_{\nu=m_1}^{n_1} a_{\nu\mu}$, $b_\nu = \sum_{m=m_2}^{n_2} a_{\nu\mu}$

$a_{\nu\mu}$: $(m_1 \le \nu \le n_1,\ m_2 \le \mu \le n_2)$

$m_1 = m_2 = 1,\ n_1, n_2 \in \mathbb{N}$:

$$\begin{array}{cccccc}
\nu & \mu & 1 & 2 & & n_2 \\
1 & & a_{11} & a_{12} & - - - - - - & a_{1n_2} \\
2 & & a_{21} & a_{22} & & a_{1n_2} \\
 & & \vert & & & \vert \\
n_1 & & a_{n_1 1} & a_{n_1 2} & a_{n_1 3} & a_{n_1 n_2}
\end{array}$$

$m_1 = m_2 = 1,\ n_1 = n_2 = n,\ 1 \le \mu \le \nu \le n$

(Zeilen addieren) $\underbrace{\sum_{\nu=1}^{n}\sum_{\mu=1}^{\nu}}_{\substack{1\le\nu\le n \\ 1\le\mu\le\nu}} a_{\nu\mu} = \underbrace{\sum_{\mu=1}^{n}\sum_{\nu=\mu}^{n}}_{\substack{1\le\mu\le n \\ \mu\le\nu\le n}} a_{\nu\mu}$ (Spalten addieren)

$$\begin{array}{cccccc}
\nu & \mu & 1 & 2 & & n \\
1 & & a_{11} & & & \\
2 & & a_{21} & a_{22} & & \\
3 & & a_{31} & a_{32} & a_{33} & \\
 & & & & & \\
n & & a_{n1} & a_{n2} & a_{n3}\ldots & a_{nn}
\end{array}
\qquad
\begin{array}{ccccc}
1 & 2 & 3 & & n \\
a_{11} & & & & \\
a_{21} & a_{22} & & & \\
a_{31} & a_{32} & a_{33} & & \\
 & & & & \\
a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn}
\end{array}$$

10.) $(0 \le \nu \le k \le n)$ $\sum_{\nu=1}^{n}\sum_{k=\nu}^{n} 1/k = \sum_{k=1}^{n}\sum_{\nu=1}^{k} 1/k = \sum_{k=1}^{n} 1/k \sum_{\nu=1}^{k} 1 = \sum_{k=1}^{n} 1/k \cdot k = \sum_{k=1}^{n} 1 = n$

S1.7.1 (901) $m \le n \in \mathbb{N}$ und Koeffizienten $a_k \in \mathbb{C}$, $m \le k \le n$.

$$\sum_{k=m, n\ge m}^{n} (a_k - a_{k+1}) = (a_m - a_{m+1}) + (a_{m+1} - a_{m+2}) + (a_{m+2} - a_{m+3}) + \ldots (a_n - a_{n+1})$$

$= a_m - a_{n+1}$ heißt Teleskopsumme

$\Delta a_k := a_k - a_{k+1}$ die erste Differenz bei a_k.

Andere Formulierung:

$$\sum_{k=m}^{n} (a_{k+1} - a_k) = a_{n+1} - a_m$$

Bsp:

1.) $\sum_{k=0}^{n} \underbrace{\left((k+1)^2 - k^2\right)}_{k^2+2k+1-k^2} = (n+1)^2 \Rightarrow (n+1)^2 = 2\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 = 2\sum_{k=0}^{n} k + \sum_{k=1}^{n} 1 + 1 = 2\sum_{k=0}^{n} k + (n+1)$

2.) $\sum_{k=0}^{n} k = 1/2((n+1)^2 - (n+1)) = 1/2(n+1)(n+1-1) = \frac{n(n+1)}{2}$

3.) $\sum\limits_{k=0}^{n} \underbrace{((k+1)^3-k^3)}_{k^3+3k^2+3k+1-k^3} = (n+1)^3 = 3\sum\limits_{k=0}^{n} k^2 + \underbrace{3\sum\limits_{k=0}^{n} k}_{\frac{3n(n+1)}{2}} + \underbrace{\sum\limits_{k=0}^{n} 1}_{n+1}$

$3\sum\limits_{k=0}^{n} k^2 = (n+1)^3 - \frac{3n(n+1)}{2} - (n+1) = (n+1)((n+1)^2 - \frac{3n}{2} - 1) =$

$(n+1)(n^2+2n+1-\frac{3n}{2}-1) = (n+1)(n^2+\frac{n}{2}) = n(n+1)\frac{2n+1}{2} \Rightarrow$

$\sum\limits_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

4.) $\sum\limits_{k=0}^{n} ((k+1)^4-k^4)$, $(n+1)^4 = \sum\limits_{k=0}^{n} k^4+4k^3+6k^2+4k+1-k^4$

$\sum\limits_{k=0}^{n} k^3 = 1/4((n+1)^4 - 6\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} - (n+1)) =$

$1/4(n+1)((n+1)^3-n(2n+1)-2n-1) = 1/4(n+1)((n+1)^3-(2n+1)(n+1)) =$

$1/4(n+1)^2((n+1)^2-(2n+1)) = \frac{1}{4}n^2(n+1)^2 = (\sum\limits_{k=1}^{n} k)^2 = \left(\frac{n(n+1)}{2}\right)^2$

5.) $\sum\limits_{k=1}^{n} \frac{1}{k(k+1} = \sum\limits_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1 - \frac{1}{n+1}$

A1.7.1

a) Berechne für $n\in\mathbb{N}$ die Summen • $\sum\limits_{j=1}^{n}\sum\limits_{k=j}^{n} \frac{j}{k}$ und •• $\sum\limits_{j=1}^{n}\sum\limits_{k=j}^{n} \frac{j^2}{k+1}$

Lös: • $\sum\limits_{j=1}^{n}\sum\limits_{k=j}^{n} \frac{j}{k} \underbrace{=}_{1\le j\le k\le n} \sum\limits_{k=1}^{n}\sum\limits_{j=1}^{k} \frac{j}{k} = \sum\limits_{k=1}^{n} \frac{1}{k}\sum\limits_{j=1}^{k} j \underbrace{=}_{Bsp\ 2.)} \sum\limits_{k=1}^{n} \frac{1}{k}\frac{k(k+1)}{2} = \sum\limits_{k=1}^{n} \frac{k(k+1)}{2k} = \sum\limits_{k=1}^{n}$

$\frac{(k+1)}{2} = \frac{1}{2}(\sum\limits_{k=1}^{n} k + \sum\limits_{k=1}^{n} 1) = \frac{1}{2}(\frac{n(n+1)}{2} + n) = \frac{n}{4}(n+1+2) = \frac{n(n+3)}{4}$

•• $\sum\limits_{k=1}^{n} \frac{1}{k+1}\sum\limits_{j=1}^{k} j^2 \underbrace{=}_{Bsp\ 3.)} \sum\limits_{k=1}^{n} \frac{k(k+1)(2k+1)}{(k+1)6} = \frac{1}{3}\sum\limits_{k=1}^{n} k^2 + \frac{1}{6}\sum\limits_{k=1}^{n} k =$

$\frac{n(n+1)(2n+1)}{18} + \frac{n(n+1)}{12} = \frac{n(n+1)}{36}(4n+2+3) = \frac{n(n+1)(4n+5)}{36}$

b) Beweise: $\sum\limits_{k=1}^{n}\left[\frac{k^3}{2}\right] = \frac{1}{2}\left(\left(\frac{n(n+1)}{2}\right)^2 - \left[\frac{n+1}{2}\right]\right)$, []: Das größte Ganze

Bew: Wenn k gerade $\Rightarrow$ k^3 gerade $\Rightarrow$ $\frac{k^3}{2}\in\mathbb{Z}$, d.h.

Wenn k ungerade $\Rightarrow$ k^3 ungerade $\Rightarrow$ $\frac{k^3}{2}-\frac{1}{2}\in\mathbb{Z}$, d.h. $\left[\frac{k^3}{2}\right] = \frac{k^3}{2}-\frac{1}{2}$.

Unter den natürlichen Zahlen 1,2,...,n sind $\left[\frac{n+1}{2}\right]$ ungerade

Zahlen $\Rightarrow$ $\sum\limits_{k=1}^{n}\left[\frac{k^3}{2}\right] = \sum\limits_{k=1}^{n}\frac{k^3}{2} - \left[\frac{n+1}{2}\right]\frac{1}{2} \underbrace{=}_{Bsp\ 4.)} \frac{1}{2}\left(\left(\frac{n(n+1)}{2}\right)^2 - \left[\frac{n+1}{2}\right]\right)$

A1.7.2 Zeige:

a) $\sum_{\nu=1}^{n} \frac{1}{\nu(\nu+1)} = 1 - \frac{1}{n+1} \quad \forall\ n \in \mathbb{N}_0$

Lös: Teleskopsumme $\sum_{k=m}^{n} (a_{k+1} - a_k) = a_{n+1} - a_m \ \ \forall\ n+1 \geq m$ (d.h. auch n+1=m !) $\Rightarrow$

$$\sum_{\nu=1}^{n} \frac{1}{\nu(\nu+1)} = \sum_{\nu=1}^{n} (1/\nu - \frac{1}{\nu+1}) = 1 - \frac{1}{n+1} \quad \text{für } n+1 \geq 1, \text{ d.h. für } n \geq 0,\ n \in \mathbb{N}_0$$

Genauer: $= \sum_{\nu=1}^{n} (\underbrace{-\frac{1}{\nu+1}}_{a_{k+1}} - (\underbrace{-\frac{1}{\nu}}_{a_k})) = a_{n+1} - a_n = -\frac{1}{n+1} - (-\frac{1}{n}) = 1 - \frac{1}{n+1}$

b) $\prod_{\nu=1}^{n} (1+1/\nu) = n+1 \quad \forall\ n \in \mathbb{N}_0$

Lös: Teleskopprodukt $\prod_{k=m}^{n} \frac{a_{k+1}}{a_k} = \frac{a_{n+1}}{a_m}$ für $n+1 \geq m$, $a_k \neq 0$ für $m \leq k \leq n$

Bew anlog Teleskopsumme, d.h. Induktion

$\Rightarrow \prod_{\nu=1}^{n} (1+1/\nu) = \prod_{\nu=1}^{n} \frac{\nu+1}{\nu} = \frac{n+1}{1} = n+1$ für $n+1 \geq 1$, d.h. $n \in \mathbb{N}$

c) $\prod_{\nu=0}^{n} (1+x^{2^\nu}) = \frac{1-x^{2^{n+1}}}{1-x} \ \ \forall\ n \in \mathbb{N}_0$ und $\forall\ x \in \mathbb{C} \setminus \{1\}$ $\left\{ \begin{matrix} 1, \textit{falls } n<m \\ a_m, \textit{falls } n=m \\ \prod_{k=m}^{n-1} a_k * a_n, \textit{falls } m+1 \geq n \end{matrix} \right\}$

(oder $\prod_{\nu=1}^{n} (1+x^{2^\nu}) = \frac{1-x^{2^{n+1}}}{1-x^2}$ oder $\prod_{\nu=1}^{n} (1+x^{2^\nu}) = \frac{1-x^{2^n}}{1-x}$ o.ä.)

*//**D1.5.2** (709) K:* $\prod_{k=m}^{n} a_k := \left\{ \begin{matrix} 1, \textit{falls } n<m \\ a_m, \textit{falls } n=m \\ \prod_{k=m}^{n-1} a_k * a_n, \textit{falls } m+1 \geq n \end{matrix} \right\}$

Bew: Induktion nach n, Induktionsanfang

n=o: $\prod_{\nu=0}^{0} (1+x^{2^\nu}) = 1 + x^{2^0} \underbrace{=}_{D\,1.5.2} 1 + x^1 = 1 + x = \frac{1-x^2}{1-x} = \frac{1-x^{2^{0+1}}}{1-x} \quad \forall\ x \neq 1$

$\underbrace{n \ a}_{n \geq 0}$ n+1: $\prod_{\nu=0}^{n+1} (1+x^{2^\nu}) = [\underbrace{\prod_{\nu=0}^{n} (1+x^{2^\nu})}_{Ind\ Hyp}] (1+x^{2^{n+1}}) \underbrace{=}_{IndHyp} \frac{1-x^{2^{n+1}}}{1-x} (1+x^{2^{n+1}}) =$

$$\frac{1-(x^{2^{n+1}})^2}{1-x} = \frac{1-x^{2 \cdot 2^{n+1}}}{1-x} = \frac{1-x^{2^{(n+1)+1}}}{1-x} \quad \forall\ x \neq 1$$

S1.7.2(904) Endliche geometrische Reihe

Vor. Seien $a,b\in\mathbb{C}$, $n\in\mathbb{N}_0$

Beh:

1.) $\sum_{k=0}^{n} a^k = \begin{cases} n+1,\ a=1 \\ \frac{1-a^{n+1}}{1-a},\ a\neq 1 \end{cases}$ 2.) $a^{n+1}-b^{n+1}=(a-b)\sum_{k=0}^{n} a^k b^{n-k}$

Bew 2.): $(a-b)\sum_{k=0}^{n} a^k b^{n-k} = \sum_{k=0}^{n} a^{k+1} b^{\overbrace{n-k}^{n+1-(k+1)}} - \sum_{k=0}^{n} a^k b^{n+1-k} =$

$$\sum_{k=1}^{n+1} a^k b^{n+1-k} - \sum_{k=0}^{n} a^k b^{n+1-k} = a^{n+1}*1 - a^0 b^{n+1} = a^{n+1}-b^{n+1}.$$

Falls $a\neq 0$ folgt mit $b=1$ 1.) aus 2.):

Bew 1.): $b=1$: $(a-1)\sum_{k=0}^{n} a^k = a^{n+1}-1$, $a\neq 0 \Rightarrow \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$

$a=1$: $\sum_{k=0}^{n} 1 = n+1$

Bsp: $a=1/2$, $\sum_{k=0}^{n} 2^{-k} = \frac{1-\overbrace{2^{-(n+1)}}^{(2^{-1})^{n+1}}}{1-1/2} = 2(1-2^{-(n+1)}) = 2-\frac{1}{2^n}$

S1.7.3(904) Abelsche partielle Summation

$m\le n\in\mathbb{N}$ und Koeffizienten $a_k, b_k\in\mathbb{C}$, $m\le k\le n-1$, $m\le \nu\le k$, $\Delta a_k = a_k - a_{k+1}$

$$\sum_{k=m}^{n} a_k b_k = a_n \sum_{k=m}^{n} b_k + \sum_{k=m}^{n-1} \Delta a_k \underbrace{\sum_{\nu=m}^{k} b_\nu}_{=:B_k} =: a_n B_n + \sum_{k=m}^{n-1} \Delta a_k B_k.$$

//***S1.7.1*** *(901)) $m\le n\in\mathbb{N}$, $a_k\in C$, $m\le k\le n$.* $\sum_{k=m, n\ge m}^{n} (a_k - a_{k+1}) = a_m - a_{n+1}$, $\Delta a_k := a_k - a_{k+1}$//

S1.7.2 *(903) $a,b\in C$, $n\in N_0$: 1.)* $\sum_{k=0}^{n} a^k = \begin{cases} n+1,\ a=1 \\ \frac{1-a^{n+1}}{1-a},\ a\neq 1 \end{cases}$

Bew: $\sum_{k=m}^{n-1} \Delta a_k \sum_{\nu=m}^{k} b_\nu = \sum_{\nu=m}^{n-1} b_\nu \sum_{k=\nu}^{n-1} (a_k - a_{k+1}) \underset{S1.7.1}{\Rightarrow} \sum_{\nu=m}^{n-1} b_\nu (a_\nu - a_n) =$

$$\sum_{\nu=m}^{n-1} a_\nu b_\nu - a_n \sum_{\nu=m}^{n-1} b_\nu + a_n b_n - a_n b_n = \sum_{\nu=m}^{n-1} a_\nu b_\nu + a_n b_n - a_n \sum_{\nu=m}^{n-1} b_\nu + a_n b_n = \sum_{\nu=m}^{n} a_\nu b_\nu - \sum_{\nu=m}^{n} a_n b_\nu.$$

Bsp: $\sum_{k=1}^{n} k2^k \underset{S1.7.3}{=} n\sum_{k=1}^{n} 2^k + \sum_{k=1}^{n-1} (-1) \sum_{\nu=1}^{k} 2^\nu = n\sum_{k=1}^{n} 2^k + \sum_{k=1}^{n-1} (-2) \sum_{\nu=1}^{k} 2^{\nu-1} =$

$$n\sum_{k=1}^{n} 2^k + \sum_{k=1}^{n-1} (-2) \sum_{\nu=0}^{k-1} 2^\nu \underset{S1.7.2}{=} n\sum_{k=1}^{n} 2^k + \sum_{k=1}^{n-1} (-2) \frac{1-2^k}{1-2} =$$

$$n\sum_{k=1}^{n} 2^k + \sum_{k=1}^{n-1} (-2)(2^k-1) = n(\sum_{k=0}^{n} 2^k - 1) - 2\sum_{k=1}^{n-1} (2^k-1) \underset{S1.7.2}{=}$$

$$n\left(\frac{1-2^{n+1}}{1-2}-1\right) - 2(\sum_{k=0}^{n-1} 2^k - 1) + 2\sum_{k=1}^{n-1} 1 \underset{S1.7.2}{=} n(\underbrace{2^{n+1}}_{2*2^n} - 2) - \frac{2(1-2^n)}{1-2} + 2 + 2(n-1) =$$

$$2n(2^n-1) - 2(2^n-1) + 2n = 2n2^n - 2n - 2*2^n + 2 + 2n = 2n2^n - 2*2^n + 2 = 2 + (n-1)2^{n+1}$$

A1.7.3 Es seien $a,b\in\mathbb{R}$ mit $a>b>0$ sowie $n\in\mathbb{N}$ gegeben.

Zeige, daß dann $na^{n-1} > \frac{a^n-b^n}{a-b} > nb^{n-1}$ gilt.

*//**S1.7.2** (903) $a,b\in C$, $n\in N_0$: 2.) $a^{n+1}-b^{n+1}=(a-b)\sum\limits_{k=0}^{n} a^k b^{n-k}$ //*

Lös: $\frac{a^n-b^n}{a-b} = \sum\limits_{k=0}^{n-1} a^k b^{n-1-k} \le \sum\limits_{k=0}^{n-1} a^k a^{n-1-k} = a^{n-1}\sum\limits_{k=0}^{n-1} 1 = na^{n-1}$ und

$\frac{a^n-b^n}{a-b} = \sum\limits_{k=0}^{n-1} a^k b^{n-1-k} \ge \sum\limits_{k=0}^{n-1} b^k b^{n-1-k} = b^{n-1}\sum\limits_{k=0}^{n-1} 1 = nb^{n-1}$.

A1.7.4 Die Folge $(a_n)_{n=1}^{\infty}$ sei definiert durch

$a_1=1$ und $a_{n+1}=\frac{1+4a_n+\sqrt{1+24a_n}}{16}$ für $n\in\mathbb{N}$. Bestimme eine explizite Darstellung von a_n.

Lös: $a_1=1$, $a_2=5/8$, $a_3=15/32$, $a_4=51/128$, $a_5=187/512$

$\sqrt{1+24a_n}$ $=5$, 4, $7/2$, $13/4$, $25/8$

Beobachtung: $2^n\sqrt{1+24a_n}=b_n$ ist ganzzahlig. Dann gilt

$b_1=10$ und für $n\in\mathbb{N}$

$b_{n+1}=2^{n+1}\sqrt{1+24a_{n+1}} = 2^{n+1}\sqrt{1+\frac{3}{2}\left(1+4a_n+\sqrt{1+24a_n}\right)} =$

$2^{n+1}\sqrt{\frac{2+3+12a_n+3\sqrt{1+24a_n}}{2}} = 2^{n+1}\sqrt{\frac{5+12a_n+3*2^{-n}b_n}{2}} =$

$2^{n+1}\sqrt{\frac{9+(1+24a_n)+6*2^{-n}b_n}{4}} = 2^n\sqrt{9+\left(\frac{b_n}{2^n}\right)^2+6*2^{-n}b_n} =$

$2^n\sqrt{\left(\frac{b_n}{2^n}+3\right)^2} = 2^n\left(\frac{b_n}{2^n}+3\right) = b_n+3*2^n \Rightarrow b_{n+1}-b_n=3*2^n \Rightarrow$

$b_n=\underbrace{\sum\limits_{k=1}^{n-1}(b_{k+1}-b_k)}_{Telesks:\,b_n-b_1} + \underbrace{10}_{b_1} = \sum\limits_{k=1}^{n-1} 3*2^k+10 = 6\sum\limits_{k=1}^{n-1} 2^{k-1}+10 \underset{\nu=k-1}{=} 6\sum\limits_{\nu=0}^{n-2} 2^\nu+10 = 6\,\frac{2^{n-1}-1}{2-1}+10 =$

$3*2^n+4$ für $n\in\mathbb{N}$.

$1+24a_n=\left(\frac{b_n}{2^n}\right)^2=\left(\frac{3*2^n+4}{2^n}\right)^2=(3+2^2 2^{-n})^2=(3+2^{2-n})^2$.

$a_n=\frac{(3+2^{2-n})^2-1}{24}$ für $n\in\mathbb{N}$

A1.7.5

a) Es seien $n\in\mathbb{N}$ und $a_1,\dots,a_n\in\mathbb{R}$. Zeige $\left(\sum\limits_{\nu=1}^{n} a_\nu\right)^2 \le n\sum\limits_{\nu=1}^{n} a_\nu^2$.

Wann genau gilt Gleichheit?

b) Zeige: $\sum\limits_{\nu=1}^{n}\frac{1}{\nu^3} < 2 \quad \forall\ n\in\mathbb{N}$.

D1.7.1(906)

Für $n \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$ sei

1.) $n! := \prod_{k=1}^{n} k = \begin{Bmatrix} 1 \text{ fir } n=0 \\ 1*2*,\ldots *n, \; n \in N \end{Bmatrix}$ heißt n Fakultät

0!=1, weil ein leeres Produkt den Wert 1 haben soll.

2.) $\left(n^{\alpha}\right) := \dfrac{\prod_{k=1}^{n}(\alpha+1-k)}{n!} = \begin{Bmatrix} 1 \text{ fir } n=0 \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}, n \in N \end{Bmatrix}$

heißt Binominalkoeffizient α über n

$\left(0^{\alpha}\right)$ =1 entsprechend 1.)

Bsp: $\left(2^{2i}\right) = \dfrac{2i(2i-1)}{2!} = \dfrac{-4-2i}{2}$ =-2-i

$\left(3^{0,1}\right) = \dfrac{0{,}1(-0{,}9)(-1{,}9)}{3!}$ =0,0285

Sonderfall $n=\alpha \in \mathbb{N}$, $\alpha<k$: $\dfrac{n(n-1)\ldots\overbrace{(n-n)}^{0}\ldots(n-k+1)}{k!}$ =0

A1.7.6

Berechne $\left(5^{1/2}\right)$ und $\left(3^{i-2}\right)$

Lös: $\dfrac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{5!} = \dfrac{1*7}{2^5*2*4} = \dfrac{7}{2^8} = \dfrac{7}{256}$

$\dfrac{(i-2)(i-3)(i-4)}{3!} = \dfrac{5}{6}$ (5-3i)

S1.7.4(906) Für $\alpha \in \mathbb{C}$ und $n,m,k \in \mathbb{N}_0$ $j \in \mathbb{N}$ gilt

1.) $\left(n^{\alpha}\right) + \left(n+1^{\alpha}\right) = \left(n+1^{\alpha+1}\right)$

Bew: $\left(n^{\alpha}\right) + \left(n+1^{\alpha}\right) =$

$$\frac{\alpha(\alpha-1)\ldots(\alpha-(n-1))}{n!} + \frac{\alpha(\alpha-1)..(\alpha-(n-1))(\alpha-n)}{(n+1)n!} =$$

$$\frac{\alpha(\alpha-1)..(\alpha-(n-1))}{n!}\underbrace{\left(1+\frac{\alpha-n}{n+1}\right)}_{\frac{n+1+\alpha-n}{n+1}} = \frac{\alpha(\alpha-1)..(\alpha-(n-1))}{n!}\left(\frac{\alpha+1}{n+1}\right) =$$

$$\frac{(\alpha+1)(\alpha+1-1)..(\alpha+1-(n+1-1))}{(n+1)n!} = \left(n+1^{\alpha+1}\right)$$

2.) $\left(m^{n}\right) = \begin{Bmatrix} 0 \text{falls } n<m \\ \dfrac{n!}{m!(n-m)!}, \text{falls } n^3 m \end{Bmatrix}$

Bew: $\left(m^{n}\right) = \dfrac{n(n-1)\ldots(n-(m-1))}{m!}$

m>n..1 Faktor im Zähler ist 0

m≤n:

$$= \frac{n(n-1)\ldots(n-(m-1))(n-m)(n-m-1)\ldots.1}{m!(n-m)!} = \frac{n!}{m!(n-m)!}$$

3.) $\left(m^n\right) = \left(n-m^n\right) \underset{2.)}{=} \dfrac{n!}{(n-m)!(n+m-n)!} = \dfrac{n!}{m!(n-m)!} \underbrace{=}_{2.)} \left(m^n\right)$ falls $n \geq m$

4.) $\dfrac{n^k}{k!}\left(1-\dfrac{k(k-1)}{2n}\right) \leq \left(k^n\right) \leq \dfrac{n^k}{k!}$, $0 \leq k \leq n$ $\qquad \dfrac{\overbrace{n(n-1)\ldots(n-(k-1))}^{kFaktoren}}{n^k}$

Bew: $\left(k^n\right) = \dfrac{n(n-1)\ldots(n-(k+1))}{k!} \leq \dfrac{n^k}{k!}$

(Ungleichung mit $\dfrac{k!}{n^k}$ multipliziert)

$$\frac{k!}{n^k}\left(k^n\right) = \frac{\overbrace{n(n-1)\ldots(n-(k-1))}^{kFaktoren}}{n^k} = 1(1-1/n)(1-2/n)\ldots\left(1-\frac{k-1}{n}\right) \geq 1-\frac{k(k-1)}{2n}$$

k $\geq$ 2: Aussage A(k) Bew durch Induktion

Induktionsanfang

k=2: $1(1-1/n) \geq 1-1/n$ richtig

$$k \to k+1: \underbrace{(1-1/n)\ldots\left(1-\frac{k-1}{n}\right)}_{\left(1-\frac{k(k-1)}{2n}\right)}(1-k/n) \underbrace{\geq}_{Ind.Hyp.} \left(1-\frac{k(k-1)}{2n}\right)(1-k/n) =$$

$$1-k/n-\frac{k(k-1)}{2n}+\frac{k^2\overbrace{(k-1)}^{0}}{\underbrace{2n^2}_{0}} \geq$$

$$1-k/n-\frac{k(k-1)}{2n} = 1-\frac{2k+k(k-1)}{2n} = 1-\frac{(k+1)k}{2n}$$

$$\Rightarrow \frac{n^k}{k!}\left(1-\frac{k(k-1)}{2n}\right) \leq \frac{n^k}{k!}\frac{k!}{n^k}\left(k^n\right) \Rightarrow \frac{n^k}{k!}\left(1-\frac{k(k-1)}{2n}\right) \leq \left(k^n\right)$$

Andere Formulierung:

$$\frac{n^k}{k!}\left(1-\frac{k(k-1)}{n}\right) \leq \left(k^n\right) \leq \frac{n^k}{k!}$$

*//**S1.5.6** (715) $x \in R$, $x \geq -1$, $n \in N$, $(1+x)^n \geq 1+nx$, $=$ $\Leftrightarrow$ $x=0$ oder $n=0$ oder $n=1$//*

$$\left(k^n\right) = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{\prod_{v=0}^{k-1}(n-v)}{k!} \leq \frac{\prod_{v=0}^{k-1} n}{k!} = \frac{n^k}{k!}$$

$$\left(k^n\right) = \frac{\prod_{v=0}^{k-1}(n-v)}{k!} \geq \frac{\prod_{v=0}^{k-1}(n-k+1)}{k!} = \frac{(n-k+1)^k}{k!} = \frac{(n(1-k/n+1/n))^k}{k!} =$$

$$\frac{n^k}{k!}\left(1-\frac{k}{n}+\frac{1}{n}\right)^k = \frac{n^k}{k!}\left(1-\frac{k-1}{n}\right)^k \underbrace{\geq}_{S1.5.6} \frac{n^k}{k!}\left(1-\frac{k(k-1)}{n}\right)$$

5.) $\dbinom{\alpha}{j}+\dbinom{\alpha}{j-1}=\dbinom{\alpha+1}{j}$

Bew:ähnlich 1.)

6.) Aus 1.) $\Rightarrow$ Binominalsatz. $\forall$ a,b,z$\in \mathbb{C}$ und n$\in \mathbb{N}_0$ gilt

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k,$$ wenn man $a^0=b^0=(a+b)^0$ setzt

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$

$$a=b=1 \Rightarrow 2^n = \sum_{\nu=0}^{n} \binom{n}{\nu}, \quad 0^0=1$$

$$a=-1, b=1 \Rightarrow \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^\nu = (1-1)^n = 0^n \quad n\in \mathbb{N}_0$$

Bew: $n=0: (a+b)^0 = 1 = 1*1*1 = \binom{0}{0} a^0 b^0 = \sum_{k=0}^{0} \binom{0}{k} a^k b^{0-k}.$

($\binom{x}{0} = 1 \quad \forall$ z$\in \mathbb{C}$ nach D1.7.1)

$$n+1: (a+b)^{n+1} = (a+b)(a+b)^n \underbrace{=}_{Ind\ Hyp} (a+b) \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \underbrace{=}_{\sum\limits_{k=0}^{n} a^k = \sum\limits_{k=0}^{n} a^{n-k}}$$

$$\sum_{k=0}^{n} \binom{n}{k} a^{1+n-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{1+k} =$$

$$\sum_{k=0}^{n} \binom{n}{k} a^{1+n-k} b^k + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n-(k-1)} b^k = \sum_{k=0}^{n} \binom{n}{k} a^{1+n-k} b^k + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{1+n-k} b^k \underbrace{=}_{5.)}$$

$$\underbrace{a^{n+1}}_{k=0\, im \sum} + \underbrace{b^{n+1}}_{k=n+1\, im \sum} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n-k+1} b^k =$$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k.$$

Andere Formulierung:

Bew: $n=0: (a+b)^0 = 1 = 1*1*1 = \binom{0}{0} a^0 b^0 = \sum_{k=0}^{0} \binom{0}{k} a^k b^{0-k}.$

$$n+1: (a+b)^{n+1} = (a+b)(a+b)^n \underbrace{=}_{Ind\ Hyp} (a+b) \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} =$$

$$\underbrace{\left[\sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k}\right]}_{=\sum\limits_{k=1}^{n+1} \binom{n}{k-1} a^k b^{n+1-k}} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} =$$

$$\left[\sum_{k=1}^{n} \left(\underbrace{\underbrace{\binom{n}{k-1} + \binom{n}{k}}_{=\binom{n+1}{k}}}_{S1.}\right) a^k b^{n+1-k}\right] + \underbrace{\binom{n}{n}}_{=1=\binom{n+1}{n+1}} a^{n+1} b^0 + \underbrace{\binom{n}{0}}_{=1=\binom{n+1}{0}} a^0 b^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}.$$

Bem: Pascalsches Dreieck $0\leq m\leq n$

n=1						1					
n=2					1		2	1			
n=3				1		3		3		1	
n=4		1		4		6		4		1	
n=5	1		5		10		10		5		1

$10=\binom{5}{3}=\frac{5*4*3}{1*2*3}=\binom{4}{3-1}+\binom{4}{3}$ entsprechend 5.)

$$\binom{\alpha+1}{j}=\binom{\alpha}{j-1}+\binom{\alpha}{j}$$

Bsp:1.) $\sum_{k=0}^{n}\left(k^{n}\right)=2^{n}$, $(1-1)^{n}=\sum_{k=0}^{n}\left(k^{n}\right)(-1)^{k}=0$

2.) $\sum_{j=0}^{n}\sum_{k=0}^{n-j}\left(k^{n}\right)\left(j^{n-k}\right)2^{k+j}=\sum_{k=0}^{n}\sum_{j=0}^{n-k}\left(k^{n}\right)\left(j^{n-k}\right)2^{k}2^{j}=\sum_{k=0}^{n}\left(k^{n}\right)2^{k}\sum_{j=0}^{n-k}\left(j^{n-k}\right)2^{j}1^{n-k-j}\underbrace{=}_{*}$

a_{jk}:

$$\begin{array}{c|cccccc}
j\ k & 0 & 1 & 2\ldots & n-2 & n-1 & n\\
0 & a_{00} & a_{01} & a_{02} & & a_{0n-1} & a_{0n}\\
1 & a_{10} & a_{11} & a_{12} & & a_{1n-1} & \\
2 & a_{20} & a_{21} & a_{22} & a_{2n-2} & & \\
. & & & & & & \\
. & & & & & & \\
n & a_{n0} & & & & &
\end{array}
\qquad
\begin{array}{c|cccccc}
j\ k & 0 & 1 & 2\ldots & n-2 & n-1 & n\\
0 & a_{00} & a_{01} & a_{02} & & a_{0n-1} & a_{0n}\\
1 & a_{10} & a_{11} & a_{12} & & a_{1n-1} & \\
2 & a_{20} & a_{21} & a_{22} & a_{2n-2} & & \\
. & & & & & & \\
. & & & a_{n-1,2} & & & \\
n & a_{n0} & & & & &
\end{array}$$

$\underbrace{=}_{*}\sum_{k=0}^{n}\left(k^{n}\right)2^{k}(2+1)^{n-k}\underbrace{=}_{**}(2+3)^{n}=5^{n}$

$*\ \left(k^{\overbrace{m}^{n-k}}\right)\underbrace{a}_{2}{}^{j}\underbrace{b}_{1}{}^{\overbrace{m}^{n-k}-j}=(\underbrace{a}_{2}+\underbrace{b}_{1})^{\overbrace{m}^{n-k}}=3^{n-k}$ $\quad **\ \sum_{k=0}^{n}\left(k^{n}\right)\underbrace{a}_{2}{}^{k}\underbrace{b}_{3}{}^{n-k}=(\underbrace{a}_{2}+\underbrace{b}_{3})^{n}$

A1.7.7 Zeige für $n\in\mathbb{N}$, $\alpha\in\mathbb{C}$, dass $\binom{\alpha}{n}=0\Leftrightarrow\alpha\in\mathbb{N}_0$ und $n>\alpha$

Lös: $\dfrac{\overbrace{\alpha(\alpha-1)..(\alpha-n+1)}^{\text{1Faktor0}}}{n!}$

A1.7.8 Zeige

a) $\sum_{v=k}^{n}\left(k^{v}\right)=\left(k+1^{n+1}\right)$ $\forall$ $k,n\in\mathbb{N}$ mit $n\geq k$

*//**S1.7.4** (906) $\alpha\in C$; $n,m,k\in N_0$, $j\in N$ 1.) $\left(n^{\alpha}\right)+\left(n+1^{\alpha}\right)=\left(n+1^{\alpha+1}\right)$ //*

Bew: Induktion nach n bei festem $k\in\mathbb{N}$

$n=k$: $\sum_{v=k}^{k}\left(k^{v}\right)=\left(k^{k}\right)=1=\left(k+1^{k+1}\right)$

$\underbrace{n\mapsto n+1}_{n\geq k}$: $\sum_{v=k}^{n+1}\left(k^{v}\right)=\left[\sum_{v=k}^{n}\left(k^{v}\right)\right]+\left(k^{n+1}\right)\underbrace{=}_{IndHyp}\left(k+1^{n+1}\right)+\left(k^{n+1}\right)\underbrace{=}_{\text{S1.7.4 1.)}}\left(k+1^{n+2}\right)=\left(k+1^{(n+1)+1}\right)$

b) $\sum_{k=0}^{n} \left(k^{n}\right)^2 = \left(n^{2n}\right)$, Hinweis: $(1+x)^{n}(1+x)^{n}=(1+x)^{2n}$

Bew: $\sum_{k=0}^{2n} \left(k^{2n}\right) x^k \underbrace{=}_{Binominalsatz} (x+1)^{2n}=(x+1)^{n}(x+1)^{n} \underbrace{=}_{Binominalsatz} [\sum_{k=0}^{n} \underbrace{\left(k^{n}\right)}_{a_k} x^k][\sum_{k=0}^{n} \underbrace{\left(k^{n}\right)}_{b_k} x^k] \underbrace{=}_{A1.9.2}$

$\sum_{k=0}^{n} \left(k^{n}\right)^2 = \left(n^{2n}\right)$, $\sum_{k=0}^{2n} [\sum_{\nu=0}^{k} \left(\nu^{n}\right) \left(k-\nu^{n}\right) x^k \ \forall \ x\in C \underbrace{\Rightarrow}_{S1.7.4} \left(k^{2n}\right) = \sum_{\nu=0}^{k} \left(\nu^{n}\right) \left(k-\nu^{n}\right) \ \forall$
k=0,1,…2n $\underbrace{\Rightarrow}_{k=n}$

$$= \sum_{\nu=0}^{n} \left(\nu^{n}\right) \underbrace{\binom{n}{\nu}}_{\underbrace{=}_{S1.7.4\,3.)}\binom{a}{b}} = \sum_{\nu=0}^{n} \left(\nu^{n}\right)^2$$

A1.7.9 Zeige für $p,q,\nu\in N_0$ die Formel $\binom{p+q}{\nu} = \sum_{j=0}^{\nu}\binom{p}{j}\binom{q}{\nu-j}$.
(siehe auch A1.9.8 (1107))
Lös: $\forall\ x\in R$ gilt $(1+x)^{p}(1+x)^{q}=(1+x)^{p+q}$ und

$$(1+x)^{p}(1+x)^{q}=\left[\sum_{j=0}^{p}\binom{p}{j}x^j\right]\left[\sum_{k=0}^{q}\binom{q}{k}x^k\right]=(1+x)^{p+q}=\left[\sum_{\nu=0}^{p+q}\binom{p+q}{\nu}x^\nu\right]$$

oBdA p>q

$$\left(\binom{p}{0}x^0+\binom{p}{1}x^1+\binom{p}{2}x^2+\binom{p}{3}x^3 +..\binom{p}{p}x^p\right)+\left(\binom{q}{0}x^0+\binom{q}{1}x^1+\binom{q}{2}x^2+\binom{q}{3}x^3+\ldots\binom{q}{q}x^q\right)=$$

$$\binom{p}{0}\binom{q}{0}x^0+\binom{p}{0}\binom{q}{1-0}x^1+\binom{p}{0}\binom{q}{2-0}x^2+\binom{p}{0}\binom{q}{3}x^3+\ldots \quad \binom{p}{0}\binom{q}{q}x^q+$$

$$\binom{p}{1}\binom{q}{1-1}x^1+\binom{p}{1}\binom{q}{2-1}x^2+\binom{p}{1}\binom{q}{2}x^3+\binom{p}{1}\binom{q}{3}x^4+\ldots \quad \binom{p}{1}\binom{q}{q-1}x^q+\binom{p}{1}\binom{q}{q}x^{q+1}+$$

$$\binom{p}{2}\binom{q}{2-2}x^2+\binom{p}{2}\binom{q}{1}x^3+\binom{p}{2}\binom{q}{2}x^4\ldots \quad +\binom{p}{2}\binom{q}{q}x^{q+2}+\ldots$$

$$\ldots \quad \ldots \quad \ldots$$

$$\binom{p}{p}\binom{q}{q}x^{q+p}$$

$= \sum_{\nu=0}^{p+q} x^\nu \sum_{j=0}^{\nu}\binom{p}{j}\binom{q}{\nu-j}$ und Koeffizientenvergl.

A1.7.10 Zeige folgende Idenditäten für Binominalkoeffizienten:
a) Folgere aus $\binom{\alpha}{k}+\binom{\alpha}{k-1}=\binom{\alpha+1}{k}$ für $\alpha\in C$ und $k\in N$, dass $\binom{n}{m}\in N_0$
$\forall$ $n,m\in N_0$.
Lös: Z.z. $\binom{n}{m}\in N_0$ $\forall$ $n,m\in N_0$. Es gilt $\binom{\alpha}{0}=1$ $\forall$ $\alpha\in C$, $\binom{n}{0}=1\in N_0$ $\forall$ $n\in N_0$,

n<m: $\binom{n}{m}=\frac{n(n-1)(n-2)..(n-m+1))(n-n)(n-n-1)\ldots(n-m+1}{m!}=$

$0\in N_0$

Induktion über n

Induktionsanfang n=0: $\binom{0}{m}=0\in N_0$ $\forall$ m>0, $\binom{0}{0}=1\in N$ und $\binom{0}{m}\in N_0$ $\forall$ $m\in N_0$.

Induktionshypothese : $\binom{n}{m}\in N_0$ $\forall$ $m\in N_0$ für ein $n\in N_0$.

Induktionsschritt $n\to n+1$: Z.z. $\binom{n+1}{m}\in N_0$ $\forall$ $m\in N_0$.

1.Fall: m=0 $\Rightarrow$ $\binom{n+1}{0}=1\in N_0$.

2.Fall:m>0 $\Rightarrow$ $\binom{n+1}{m} = \underbrace{\binom{n}{m}}_{\in N_0} + \underbrace{\binom{n}{m-1}}_{\text{IH:}\in N_0} \in \mathbb{N}_0 \Rightarrow$ Beh.

b) $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0 \quad \forall\ n \in \mathbb{N}$.

Bew: $= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k} = (-1+1)^n = 0^n = 0$

c) $\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \binom{n+1}{2} \quad \forall\ n \in \mathbb{N}$.

Bew: Induktionsanfang n=1 : $(-1)^1 1^2 = -1$, $(-1)^1 \binom{2}{2} = -1$

Induktionshypothese : $\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \binom{n+1}{2} \quad \forall\ n \in \mathbb{N}$.

Induktionsschritt n→n+1: Z.z. $\sum_{k=1}^{n+1} (-1)^k k^2 = (-1)^{n+1} \binom{n+2}{2}$.

$$\sum_{k=1}^{n+1} (-1)^k k^2 = \sum_{k=1}^{n} (-1)^k k^2 + (-1)^{n+1} (n+1)^2 \underbrace{=}_{IndH}$$

$$(-1)^n \binom{n+1}{2} + (-1)^{n+1} (n+1)^2 =$$

$$(-1)^{n+1} \left(-\binom{n+1}{2} + (n+1)^2\right) = (-1)^{n+1} \binom{n+2}{2}$$

NR: $(n+1)^2 - \binom{n+1}{2} = n^2 + 2n + 1 - \frac{(n+1)n}{2!} =$

$\frac{2n^2+4n+2-n^2-n}{2} = \frac{(n+2)(n+1)}{2} = \binom{n+2}{2}$

A1.7.11 Beweise für n∈ℕ die Ungleichung $\frac{1}{2n+1} \le \frac{1}{4^n} \binom{2n}{n} \le \frac{1}{\sqrt{2n+1}}$

durch vollständige Induktion.

//***S1.7.4*** *(906) n,m∈N₀, j∈N: 2.)* $\binom{n}{m} = \begin{cases} 0 \text{ falls } n<m \\ \frac{n!}{m!\,(n-m)!}, \text{ falls } n \ge m \end{cases}$ // $\binom{2n+2}{n+1}$

Bew: n=1: $1/3 \le \frac{1}{4} \binom{2}{1} \le \frac{1}{\sqrt{3}} \Leftrightarrow 1/3 \le \frac{1}{2} \le \frac{1}{\sqrt{3}}$

n↦n+1: Beachte, dass $\frac{1}{4^{n+1}} \binom{2(n+1)}{n+1} = \frac{1}{4^{n+1}} \binom{2n+2}{n+1} \underbrace{=}_{S1.7.4\,2.)} \frac{1}{4^{n+1}} \frac{(2n+2)!}{((n+1)!)^2} =$

$$\frac{1}{4^n} \frac{2n!}{(n!)^2} \frac{1}{4} \frac{(2n+1)(2n+2)}{(n+1)^2} = \frac{1}{4^n} \frac{2n!}{(n!)^2} \frac{(2n+1)(n+1)}{(2(n+1))^2} \underbrace{=}_{S1.7.4\,2.)} \frac{2n+1}{2n+2} \underbrace{\frac{1}{4^n}\binom{2n}{n}}_{I.H. \ge \frac{1}{2n+1}} \ge$$

$$\frac{2n+1}{2n+2} \frac{1}{2n+1} \ge \frac{1}{2(n+1)} \ge \frac{1}{2(n+1)+1} \quad \text{und}$$

$$\frac{1}{4^{n+1}} \binom{2n+2}{n+1} = \frac{2n+1}{2n+2} \underbrace{\frac{1}{4^n}\binom{2n}{n}}_{\frac{1}{\sqrt{2n+1}}} \le \frac{2n+1}{2n+2} \frac{1}{\sqrt{2n+1}} ==$$

$$\frac{1}{\sqrt{2n+3}} \frac{\sqrt{(2n+3)(2n+1)}}{\sqrt{(2n+2)^2}} = \frac{1}{\sqrt{2n+3}} \sqrt{\frac{4n^2+8n+3}{4n^2+8n+4}} \le \frac{1}{\sqrt{2n+3}}$$

A1.7.13 Zu Zeigen: $n!>\left(\frac{n}{2}\right)^{\frac{n}{2}}\quad \forall\ n\in\mathbb{N}$

#**S1.5.6**(715)Unleichung von Bernoulli
#Vor:$x\in\mathbb{R},\ x\geq -1,\ n\in\mathbb{N}$
#Beh:$(1+x)^n\geq 1+nx$

Lös:IA: n=1: $1!=1>\frac{1}{\sqrt{2}}=\left(\frac{1}{2}\right)^{\frac{1}{2}}$

IH: $n!>\left(\frac{n}{2}\right)^{\frac{n}{2}}$ gelte für ein $n\in\mathbb{N}$

IS: $(n+1)!=\underset{\geq 0}{(n+1)}\,n!>(n+1)\left(\frac{n}{2}\right)^{\frac{n}{2}}\geq\left(\frac{n+1}{2}\right)\left(\frac{n}{2}\right)^{\frac{n}{2}}\geq\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^{\frac{n}{2}}\geq\left(\frac{n}{2}\right)^{\frac{1}{2}}\left(\frac{n}{2}\right)^{\frac{n}{2}}=\left(\frac{n+1}{2}\right)^{\frac{n+1}{2}}$

$\Leftrightarrow (n+1)^2\left(\frac{n}{2}\right)^n\geq\left(\frac{n+1}{2}\right)^{n+1}\quad\Leftrightarrow\quad\left(\frac{\frac{n}{2}}{\frac{n+1}{2}}\right)^{\frac{n}{2}}\geq\frac{\frac{n+1}{2}}{(n+1)^2}\quad\Leftrightarrow$

$\left(\frac{n}{n+1}\right)^n=(1-\underset{\geqslant -1}{\frac{1}{n+1}})^n\underset{S1.5.6}{\geqslant}1-\frac{n}{n+1}=\frac{1}{n+1}\geq\frac{1}{2(n+1)}$

A1.7.14 Angeordneter Körper

$\forall\ x\in\mathbb{K},\ a\geq 0,\ (1+a)^n\geq\frac{n^2}{4}a^2$

Lös: $(1+a)^n=\sum_{k=0}^{n}\binom{n}{k}a^k\underbrace{\geqslant}_{k\geqslant 2}\binom{n}{2}a^2$ da $\binom{n}{k}a^k\geq 0\quad\forall\ 0\leq k\leq n.$

$\binom{n}{2}=\frac{n(n-1)}{2}\geq\frac{n^2}{4}$, denn $n-1\underbrace{\geqslant}_{n\geqslant 2}\frac{n}{2}\Leftrightarrow 2n-2\geq n\ \Leftrightarrow\ n\geq 2$

$\left(\binom{n}{2}=\frac{n!}{(n-2)!2!}=\frac{1*2\ldots(n-2)(n-1)n}{1*2*\ldots(n-2)!2}\right)$